ISAS/32/2004/FM

Semiclassical Scaling Functions of Sine–Gordon Model

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Abstract

We present an analytic study of the finite size effects in Sine–Gordon model, based on the semiclassical quantization of an appropriate kink background defined on a cylindrical geometry. The quasi–periodic kink is realized as an elliptic function with its real period related to the size of the system. The stability equation for the small quantum fluctuations around this classical background is of Lamé type and the corresponding energy eigenvalues are selected inside the allowed bands by imposing periodic boundary conditions. We derive analytical expressions for the ground state and excited states scaling functions, which provide an explicit description of the flow between the IR and UV regimes of the model. Finally, the semiclassical form factors and two-point functions of the basic field and of the energy operator are obtained, completing the semiclassical quantization of the Sine–Gordon model on the cylinder.

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1 Introduction

Quantum field theory on a finite volume is a subject of both theoretical and practical interest. It almost invariably enters the extrapolation procedure of numerical simulations, limited in general to rather small samples, but it is also intimately related to quantum field theory at finite temperature. It is therefore important to increase our ability in treating finite size effects by developing efficient analytic means. In the last years, a considerable progress has been registered in particular on the study of finite size behaviour of two dimensional systems. Also for these models, however, an exact treatment of their finite size effects has been obtained only in particular situations, namely when the systems are at criticality or if they correspond to integrable field theories. At criticality, in fact, methods of finite size scaling and Conformal Field Theory [1, 2] permit to determine many universal amplitudes and to extract as well useful information on the entire spectrum of the transfer matrix. Away from criticality, exact results can be obtained only for those integrable theories described by a factorized and elastic scattering matrix [3, 4] which, on a finite volume, can be further analysed by means of Thermodynamical Bethe Ansatz [5–9]. This technique provides integral equations for the energy levels, mostly solved numerically. In all other cases, the control of finite size effects in two dimensional QFT has been reached up to now either by conformal perturbation theory or numerical methods as, for instance, the one proposed in [11].

The aim of this paper is to study the finite size effects of a two dimensional massive QFT by using a different approach, i.e. the non–perturbative semiclassical expansion formulated in the infinite volume case by Dashen, Hasslacher, Neveu [12] and by Goldstone and Jackiw [13]. Apart from some issues which make such an analysis an interesting subject in itself, the main theoretical motivation of this work consists in the possibility of obtaining analytic results for the form factors and the energy levels at a finite geometry. In integrable cases, this adds to the above techniques (see also [10]), whereas for non–integrable models it is an efficient alternative to perturbative or numerical studies. As a matter of fact, in the infinite volume case, semiclassical methods have proved to be, together with Form Factor Perturbation Theory [16], ideal tools in the analysis of non–integrable quantum field theories (see, for instance, Ref. [17]).

Form factors at a finite volume of local operators in both integrable and non–integrable theories have been studied in one of our previous papers [18]. These quantities enter the spectral density representation of correlation functions which need, however, another set of data for their complete determination, precisely the energies of the intermediate states at a finite volume. This paper is mainly devoted to fill this gap, that is, to face the problem of a semiclassical computation of the energies $E_i(R)$ of vacua and excited states as functions of the circumference R of a cylindrical geometry. Notice that, isolating a

factor $1/R$ in front of the $E_i(R)$'s (simply due to their dimensionality), the remaining quantities are *scaling functions* of the variable $r = mR$, where m is the lowest mass of the considered QFT.

It is worth to underline an important feature that has come out from the study of the semiclassical form factors in infinite volume. As we will discuss later, their accuracy seems to extend, somehow, beyond the regime in which they were supposed to be valid. Together with the known fast convergency properties of the spectral series and the information that can be extracted on energy levels, the above result suggests that the semiclassical method may provide a rather precise estimate of finite volume correlation functions, an outcome which may be useful for many applications.

For methodological reasons, we have decided to present the semiclassical computation of finite volume energies for a system that admits one of the simplest analysis, the Sine–Gordon (SG) model. As we will see, this model is particularly appealing for its simplified semiclassical results whereby the significant physical effects we are looking for will not be masked by other additional complications. Moreover, due to the integrable nature of this theory, its finite size effects have been previously studied by means of Thermodynamical Bethe Ansatz [8, 9], and it would be interesting to perform a quantitative comparison between these results and the semiclassical ones, in order to directly control their range of validity. However, as already pointed out, semiclassical methods apply not only to integrable theories and this opens the way to describe analytically the finite size effects also in non–integrable models [19].

The paper is organized as follows: in Section 2 we briefly recall the main ideas and results of the semiclassical approach. We also discuss the simplest scaling function in a finite volume in order to clarify the nature of divergencies encountered in such computations. Section 3 is devoted to the complete semiclassical analysis of the energies of the quantum states in the kink sector of the SG model on a cylinder. In general, this analysis passes through the solution of a Schrödinger type equation for a particle in a periodic potential and, for the SG model, this corresponds to a Lamé differential equation. In this section we also discuss how to select the proper eigenvalues inside the band structure of the spectrum in order to determine the energy levels $E_i(R)$. In Section 4 we compute the form factors of local operators by using the semiclassical methods and we comment on their properties. Our conclusions and further directions are discussed in Section 5. There are also several appendices: Appendix A presents the quantization of a free bosonic theory in a finite volume and a comparison of finite–volume and finite–temperature computations of the simplest one–point correlation function. Appendix B collects relevant mathematical properties of the elliptic functions used in the text whereas Appendix C displays the main properties of the Lamé equation.

2 Semiclassical quantization

In this section, after recalling the basic equations of the semiclassical quantization, we will present the simplest example of a scaling function on a cylindrical geometry, i.e. the ground state energy $\mathcal{E}_0^{\rm vac}(R)$ of a free massive bosonic field. In a semiclassical quantization, $\mathcal{E}_0^{\rm vac}(R)$ is the lowest energy level in the vacuum sector of the theory. This example will show, in particular, how to handle the divergencies usually encountered in the calculation of the scaling functions.

2.1 DHN method

The main feature of a large class of 2-D field theories with non–linear interaction and discrete degenerate minima is that they admit non–perturbative finite–energy classical solutions (called kinks or solitons) carrying topological charges $Q^{\pm}_{\rm top} = \pm 1$. In this paper we will concentrate our attention, in particular, on a specific model of this kind, i.e. the Sine–Gordon (SG), defined by the potential

$$V_{SG}(\phi) \;=\; \frac{m^2}{\beta^2}\,(1-\cos\beta\phi)\quad . \tag{2.1}$$

In such theories, the kinks generally interpolate between two next neighbouring minima of the potential (vacua) which are constant solutions of the equation of motion (in our example $\phi_{SG} = \frac{2\pi s}{\beta}$, $s = 0, 1$), and they exhibit certain particle properties. For instance, they are localized and topological stable objects, i.e. they do not decay into mesons with $Q_{\rm top} = 0$. Moreover, in integrable theories as SG model, their scattering is dispersionless and, in the collision processes, they preserve their form simply passing through each other.

The kinks are static solutions of the equation of motion, i.e. they are time independent in their rest frame, and they can be simply obtained by integrating the first order differential equation

$$\frac{1}{2}\left(\frac{\partial\phi_{cl}}{\partial x}\right)^2 \;=\; V(\phi_{cl}) + A\ \ , \tag{2.2}$$

further imposing that $\phi_{cl}(x)$ reaches two different minima of the potential $V(\phi)$ at $x \to \pm\infty$. These boundary conditions, which describe the infinite volume case, require the vanishing of the integration constant A. As we will see in the next Section, the kink solutions in a finite volume correspond instead to a non–zero value of A, related to the size R of the system.

All the above properties of the kink solutions are an indication that they can survive the quantization, giving rise to the quantum states in one–particle sector of the corresponding QFT. A direct correspondence among the kink states and the corresponding classical solution has been established by Goldstone and Jackiw who have shown in

Ref. [13] that the matrix element of the field ϕ between kink states is given, at leading order in the semiclassical limit, by the Fourier transform of the kink background. We will discuss this result and its applications in Sect. 4.

At the moment we are mainly concerned with the semiclassical quantization of the small fluctuations around kink backgrounds. As it is well known, one cannot apply directly to them the standard perturbative methods of quantization around the free field theory since the kinks are entirely non–perturbative solutions of the interacting theory. Their classical mass, for instance, is usually inversely proportional to the coupling constant (in the SG model one has $M_{cl} = \frac{8m}{\beta^2}$). In an infinite volume, an effective method for the semiclassical quantization of such kink solutions (as well as of the vacua ones) has been developed in a series of papers by Dashen, Hasslacher and Neveu (DHN) [12] by using an appropriate generalization of the WKB approximation in quantum mechanics (see also Refs. [14] or [15] for a review). The DHN method consists in initially splitting the field $\phi(x,t)$ in terms of its classical static solution of the equation of motion (which can be either one of the vacua or the kink configuration) and its quantum fluctuations, i.e.

$$\phi(x,t) \,=\, \phi_{cl}(x) + \eta(x,t) \quad , \quad \eta(x,t) \,=\, \sum_k e^{i\omega_k t}\, \eta_k(x) \ , \tag{2.3}$$

and in further expanding the Hamiltonian of the theory in powers of η, by keeping only the quadratic terms. As a result of this procedure, $\eta_k(x)$ satisfies the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + V''(\phi_{cl})\right] \eta_k(x) \,=\, \omega_k^2\, \eta_k(x) \ , \tag{2.4}$$

together with certain boundary conditions. The semiclassical energy levels in each sector are then built in terms of the energy of the corresponding classical solution and the eigenvalues ω_i of the stability equation (2.4), i.e.

$$E_{\{n_i\}} \,=\, E_{cl} + \hbar \sum_k \left(n_k + \frac{1}{2}\right) \omega_k + O(\hbar^2) \ , \tag{2.5}$$

where n_k are non–negative integers. In particular the ground state energy in each sector is obtained by choosing all $n_k = 0$ and it is therefore given by[1]

$$E_0 \,=\, E_{cl} + \frac{\hbar}{2} \sum_k \omega_k + O(\hbar^2) \ . \tag{2.6}$$

In summary, to each static finite energy solution of (2.2) corresponds a tower of quantum energy eigenstates (2.5) representing the 0–particle (vacua) and 1–particle (kink),

[1] From now on we will fix $\hbar = 1$, since it is well known that the semiclassical expansion in $\hbar$ is equivalent to the expansion in the interaction coupling.

and their excitations. The construction of the complete Hilbert space, including the n–particle sectors (for $n \geq 2$), requires to consider time–dependent multi–kink and breather solutions with finite energy. Their semiclassical quantization can be performed with an appropriate modification of the DHN method [12].

As we have mentioned in the Introduction, the analytic form of the semiclassical scaling functions for the 2-D QFT's admitting static kink solutions can be achieved by DHN method suitably adapted to finite size geometry. In the following we will discuss in details the results of Sine–Gordon model on a cylindrical geometry which, as we shall see, admits the simplest technical analysis. On this geometry — described by a space variable compactified on a circle of circumference R and by a time variable t running on an infinite interval — the SG model admits quasi–periodic boundary conditions (b.c.)

$$\phi(x+R,t) \;=\; \phi(x,t) + \frac{2n\pi}{\beta} \;\;, \tag{2.7}$$

where the arbitrary winding number $n \in Z$ originates from the invariance of the potential (2.1) under $\phi \to \phi + \frac{2n\pi}{\beta}$. In particular, since we are interested in the one–kink sector, which is defined by $n = 1$, we will impose the b.c.

$$\phi(x+R,t) \;=\; \phi(x,t) + \frac{2\pi}{\beta} \;\;. \tag{2.8}$$

The first step for applying the semiclassical method to this problem is to find the finite size analog of the kink solution, satisfying now the b.c.'s (2.8). However, the success in constructing the scaling functions depends on whether one is able to solve the corresponding Schrödinger equation (2.4) and to derive an analytical expression for its frequencies ω_k. It turns out that the semiclassical finite size effects in SG model are intrinsically related to the simplest ($N = 1$) Lamè equation, which admits a complete analytical study.

2.2 SG in infinite volume

The semiclassical quantization of the Sine–Gordon soliton in infinite volume has been performed in [12]. We report here the basic results in order to show how the semiclassical technique works in the simplest example and also to introduce the quantities that should be obtained in the IR limit of the forthcoming finite volume results.

The classical (anti)soliton

$$\phi_{cl}(x) = \frac{4}{\beta} \arctan\left(e^{\pm m(x-x_0)}\right) \tag{2.9}$$

is solution of eq. (2.2) with $A = 0$. It connects the two degenerate vacua $\phi = 0$ and $\phi = \frac{2\pi}{\beta}$ and its classical mass is given by $M_{\rm cl} = 8\frac{m}{\beta^2}$. Plugging the above expression in (2.4), this

equation can be cast in the hypergeometric form by using the variable $z = \frac{1}{2}(1+\tanh mx)$, and its solution is expressed as

$$\eta(x) \;=\; \frac{1}{\beta}\, z^{\frac{1}{2}\sqrt{1-\frac{\omega^2}{m^2}}}\,(1-z)^{-\frac{1}{2}\sqrt{1-\frac{\omega^2}{m^2}}}\, F\left(2,-1,1+\sqrt{1-\frac{\omega^2}{m^2}},z\right) \;. \tag{2.10}$$

The corresponding spectrum is given by the discrete value $\omega_0^2 = 0$ (i.e. the zero mode associated to the translation invariance of the theory) and by the continuous part $\omega_q^2 = m^2(1+q^2)$, characterized by the absence of reflection and by the phase shift $\delta(q) = 2\arctan(\frac{1}{q})$.

The semiclassical correction to the mass is given by the difference between the ground state energy of the soliton and the one of the vacuum, with the addition of a mass counterterm due to normal ordering of the interaction term in the Hamiltonian:

$$M - M_{\rm cl} \;=\; \frac{1}{2}\sum_n \left[m\sqrt{1+q_n^2} - \sqrt{k_n^2+m^2}\right] - \frac{\delta\mu^2}{\beta^2}\int\limits_{-\infty}^{\infty} dx\,[1-\cos\beta\phi_{cl}(x)] \;, \tag{2.11}$$

with

$$\delta\mu^2 \;=\; -\frac{m^2\beta^2}{8\pi}\int\limits_{-\infty}^{\infty}\frac{dk}{\sqrt{k^2+m^2}} \;. \tag{2.12}$$

The discrete values q_n and k_n are obtained by defining the system in a large finite volume of size R with periodic boundary conditions:

$$2n\pi \;=\; k_n R \;=\; m\, q_n\, R + \delta(q_n) \;. \tag{2.13}$$

Sending $R \to \infty$ and computing the integrals, one finally obtains the semiclassical quantum correction to the mass of the kink

$$M \;=\; \frac{8m}{\beta^2} - \frac{m}{\pi} \;. \tag{2.14}$$

As it is well known, the exact solution of the quantum Sine–Gordon model shows that the coupling constant β^2 renormalises as [20]

$$\beta^2 \to \gamma \;=\; \frac{\beta^2}{1-\beta^2/8\pi} \;. \tag{2.15}$$

Moreover, the exact quantum mass of the soliton is given by $M = \frac{8m}{\gamma}$, which coincides with the above expression (2.14). The equality of the semiclassical and the exact result for the soliton mass is a remarkable property of the SG model, on which we will come back in the following Sections.

2.3 Ground state energy regularization in finite volume

As shown by eq. (2.6), quantum corrections to energy levels are given by the series on the frequencies ω_n. However, this series is generally divergent (this is the usual UV divergence in field theory) and a criterion is needed to regularize it. It is quite instructive to consider the simplest example where such divergence occur, i.e. in the calculation of the ground state energy $\mathcal{E}_0^{\text{vac}}(R)$ of the vacuum sector of the theory on a cylindrical geometry of circumference R. This can be constructed by implementing the DHN procedure for one of the constant solutions, for instance $\phi_{\text{cl}}^{\text{vac}} = 0$, imposing periodic boundary conditions for the corresponding fluctuations $\eta^{\text{vac}}(x)$. Obviously, what comes out is nothing else but the Casimir energy of a free bosonic field $\phi(x,t)$ with mass m. In this case the frequency eigenvalues are fixed to be

$$\omega_n \,=\, \sqrt{p_n^2 + m^2} \;, \tag{2.16}$$

with $p_n = 2\pi n/R$ and $n = 0, \pm 1, \pm 2, \ldots$.

The ground state energy has to be regularized by subtracting its infinite–volume continuous limit: this ensures in fact the proper normalization of this quantity, expressed by

$$\lim_{R\to\infty} \mathcal{E}_0^{\text{vac}}(R) \,=\, 0 \;. \tag{2.17}$$

The ground state energy at a finite volume is therefore defined by

$$\mathcal{E}_0^{\text{vac}}(R) \,=\, \frac{1}{2}\sum_{n=-\infty}^{\infty} \sqrt{\left(\frac{2\pi n}{R}\right)^2 + m^2} - \frac{1}{2}\int_{-\infty}^{\infty} dn\, \sqrt{\left(\frac{2\pi n}{R}\right)^2 + m^2} \;. \tag{2.18}$$

Isolating the zero mode, it can be conveniently rewritten as

$$\mathcal{E}_0^{\text{vac}}(R) \,=\, \frac{m}{2} + \frac{2\pi}{R}\sum_{n=1}^{\infty} \sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2} - \frac{2\pi}{R}\int_0^{\infty} dn\, \sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2} \,, \tag{2.19}$$

where $r \equiv mR$. Since the divergence of the series is due to the large n behaviour of the first two terms in the expansion

$$\sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2} \,\simeq\, n + \frac{1}{2}\left(\frac{r}{2\pi}\right)^2 \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \,, \tag{2.20}$$

we begin our calculation by subtracting and adding these divergent terms to it:

$$\begin{aligned} S(r) \,\equiv\, \sum_{n=1}^{\infty} \sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2} \,&=\, \sum_{n=1}^{\infty} \left\{ \sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2} - n - \frac{1}{2}\left(\frac{r}{2\pi}\right)^2 \frac{1}{n} \right\} + \\ &+\, \sum_{n=1}^{\infty} n + \frac{1}{2}\left(\frac{r}{2\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n} \;. \end{aligned} \tag{2.21}$$

The first series in the right hand side of the above expression is now convergent, whereas the last two terms should be coupled to the analogous ones coming from the integral, whose divergencies have to be handled in strict correspondence with those coming from the series. Hence, by subtracting and adding the leading divergence to the integral

$$
\begin{aligned}
I(r) \;&\equiv\; \int_0^\infty dn \sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2} = \\
&= \int_0^\infty dn \left\{ \sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2} - n \right\} + \int_0^\infty dn \, n \, ,
\end{aligned}
\tag{2.22}
$$

we can combine the last term in this expression with the one in (2.21) and implement the well known regularization

$$
\sum_{n=0}^{\infty} n - \int_0^\infty n \, dn \;=\; \lim_{\alpha \to 0} \left[\sum_{n=0}^{\infty} n \, e^{-\alpha n} - \int_0^\infty n \, e^{-\alpha n} \, dn \right] \;=\; -\frac{1}{12} \; . \tag{2.23}
$$

However, the first term in (2.22) still contains a subleading logarithmic divergence, as it can be seen by explicitly computing the integral by using a cut-off Λ, in the limit $\Lambda \to \infty$

$$
\int_0^\Lambda dn \left\{ \sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2} - n \right\} = \frac{1}{2} \left(\frac{r}{2\pi}\right)^2 \ln 2\Lambda + \frac{1}{4} \left(\frac{r}{2\pi}\right)^2 - \frac{1}{2} \left(\frac{r}{2\pi}\right)^2 \ln \frac{r}{2\pi} \; . \tag{2.24}
$$

This divergence can be cured by subtracting and adding the term $\frac{1}{2}\left(\frac{r}{2\pi}\right)^2 \ln \Lambda$. By combining this last term with its analogous in the series we have

$$
\lim_{\Lambda \to \infty} \left(\sum_{n=1}^{\Lambda} \frac{1}{n} - \ln \Lambda \right) \;=\; \gamma_E \;\; , \tag{2.25}
$$

where γ_E is the Euler-Mascheroni constant, while the remaining part of (2.24) with the above subtraction is now finite.

Collecting the above results, the finite expression of the ground state energy on a cylinder is then given by

$$
\mathcal{E}_0^{\text{vac}}(R) \;=\; \frac{1}{R} \left[-\frac{\pi}{6} + \frac{r}{2} + \frac{r^2}{4\pi} \left(\ln \frac{r}{4\pi} + \gamma_E - \frac{1}{2} \right) + \sum_{n=1}^{\infty} \left(\sqrt{(2\pi n)^2 + r^2} - 2\pi n - \frac{r^2}{4\pi n} \right) \right] . \tag{2.26}
$$

It is now easy to see that (2.26) fulfills the requirement of modular invariance of the theory, which imposes its equality with the TBA expression [7]

$$
\mathcal{E}_0^{\text{vac}}(R) \;=\; -\frac{\pi c(r)}{6R} \; , \tag{2.27}
$$

where

$$c(r) \;=\; -\frac{6r}{\pi^2}\int_0^\infty d\theta\,\cosh\theta\,\ln\left(1-e^{-r\cosh\theta}\right)\;. \tag{2.28}$$

In fact, this integral formula can be expressed in terms of Bessel functions, which admit a series representation that directly leads to (2.26) (see Ref. [7]). For this theory we obviously have $c(0) = 1$. Moreover, one can also check that the above regularization scheme ensures the agreement between the R and L channel calculations of the finite expression of the one–point functions $\langle\phi^{2k}\rangle$ [21]. The interested reader can find the simplest example of these calculations in Appendix A.

Finally, it is worth to note that the result (2.26) can also be obtained by using a simpler prescription which automatically includes the subtraction of the various divergencies, fastening the calculation. This consists in ignoring the divergent part of the integral, keeping only its finite part, and in regularizing the divergent series as

$$\left.\sum_{n=1}^{\infty} n\right|_{\text{reg}} \;=\; -\frac{1}{12}\;, \tag{2.29}$$

$$\left.\sum_{n=1}^{\infty} \frac{1}{n}\right|_{\text{reg}} \;=\; \gamma_E + \ln\frac{r}{2\pi}\;. \tag{2.30}$$

Formula (2.29) is the standard regularization of the Riemann zeta function $\zeta(-1)$, where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$, and usually corresponds to normal ordering with respect to the infinite volume vacuum (see, for instance, [22], chapter 4). On the contrary, the regularization of the second series is a–priori ambiguous due to its logarithmic divergence, and its finite value (2.30) was chosen according to the above discussion.

3 Scaling functions on the cylinder

We will now develop a complete semiclassical scheme to analyse the energy of the quantum state in SG model containing one soliton on the cylinder. This can be achieved by applying the DHN method to an appropriate kink background.

3.1 Properties of the periodic kink solution

In order to identify a kink on the cylinder, we have to look for a static finite energy solution of the SG model satisfying the quasi–periodic boundary condition (2.8). For the first order equation

$$\frac{1}{2}\left(\frac{\partial\phi_{cl}}{\partial x}\right)^2 \;=\; \frac{m^2}{\beta^2}\left(1-\cos\beta\phi_{cl}+A\right) \tag{3.1}$$

a solution which has this property can be found for $A > 0$. It can be expressed as

$$\phi_{cl}(x) = \frac{\pi}{\beta} + \frac{2}{\beta}\,\mathrm{am}\left(\frac{m(x-x_0)}{k}, k^2\right) , \qquad k^2 = \frac{2}{2+A} , \tag{3.2}$$

provided the circumference R of the cylinder is identified with

$$R = \frac{2}{m}\, k\, \mathbf{K}\left(k^2\right) , \tag{3.3}$$

where $\mathbf{K}(k^2)$ denotes the complete elliptic integral of the first kind[2]. The parameter x_0 in (3.2) represents the kink's center of mass position, and its arbitrariness is due to the translational invariance of the theory around the cylinder axis. The behaviour of (3.2) as a function of the real variable x is shown in Figure 1.

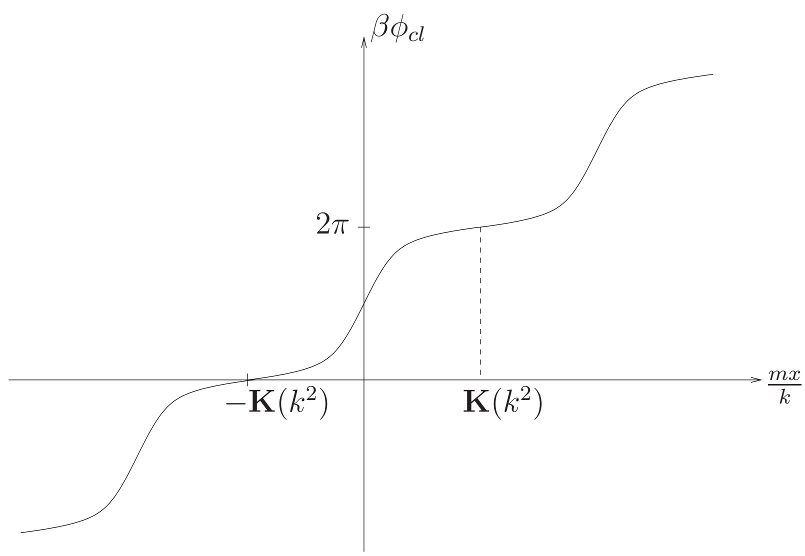

Figure 1: Solution of eq. (3.1) with $A > 0$ and $x_0 = 0$.

The function (3.2) has been first proposed in [23] and interpreted as a crystal of solitons in the sine-Gordon theory in infinite volume. In our finite volume case, instead, (3.2) has to be seen as a single soliton defined on a cylinder of circumference R (given by eq. (3.3)), while its quasi-periodic oscillations represent winding around the cylinder. As shown in eq. (3.3), there is an explicit relation between the size of the system and the integration constant A. It is easy to see that the infinite volume solution (2.9) is consistently recovered from (3.2) in the limit $A \to 0$, i.e. when R goes to infinity.

[2]The definition and basic properties of $\mathbf{K}(k^2)$ and the Jacobi elliptic amplitude $\mathrm{am}(u, k^2)$ can be found in Appendix B.

The classical energy of the kink on the cylinder is given by

$$\mathcal{E}_{cl}(R) \,=\, \int\limits_{-R/2}^{R/2} dx \left[\frac{1}{2}\left(\frac{\partial\phi_{cl}}{\partial x}\right)^2 + \frac{m^2}{\beta^2}(1-\cos\beta\phi_{cl})\right] \,=\, \frac{8m}{\beta^2}\left[\frac{\mathbf{E}(k^2)}{k} + \frac{k}{2}\left(1-\frac{1}{k^2}\right)\mathbf{K}(k^2)\right] \,, \tag{3.4}$$

where $\mathbf{E}(k^2)$ is the complete elliptic integral of the second kind. In the $R \to \infty$ limit (which corresponds to $k' \to 0$, with $(k')^2 \equiv 1 - k^2$), $\mathcal{E}_{cl}(R)$ approaches exponentially the correct value $M_\infty = \frac{8m}{\beta^2}$. This can be seen expanding $\mathbf{E}$ and $\mathbf{K}$ for small k' (see Appendix B), and expressing the result in terms of mR, which can be itself expanded in k' in virtue of the relation (3.3):

$$e^{-mR} \,=\, \frac{1}{16}(k')^2 + \cdots \,.$$

Hence the large R expansion of the classical energy is

$$\mathcal{E}_{cl}(R) \,=\, M_\infty + \frac{32}{\beta^2}\, m\, e^{-mR} + O\left(e^{-2mR}\right) \,. \tag{3.5}$$

We will comment more on the interpretation of this result in Section 3.3.

Similarly, one can derive the behaviour of $\mathcal{E}_{cl}(R)$ for small $r = mR$, which corresponds to the limit $A \to \infty$ (or $k^2 \to 0$):

$$\mathcal{E}_{cl}(R) \,=\, \frac{2\pi}{R}\,\frac{\pi}{\beta^2} + m\frac{r}{\beta^2} - m\left(\frac{r}{2\pi}\right)^3\,\frac{\pi}{2\beta^2} + \cdots \tag{3.6}$$

This formula will be relevant in the discussion of the UV properties of the scaling functions presented in Sect.3.3.

Before moving to the quantization of the kink–background (3.2), it is worth to mention that another simple kind of elliptic function, which solves eq. (3.1) for $-2 < A < 0$, was also proposed in [23] and interpreted as a crystal of solitons and antisolitons in the infinite volume SG. This background corresponds as well to a kink on the cylinder geometry but satisfying the *antiperiodic* boundary conditions

$$\phi(x+R,t) \,=\, \frac{2\pi}{\beta} - \phi(x,t) \,.$$

The associated form factors were obtained in [18]. Although the quantization of this second kink solution is technically similar to the one of (3.2) presented in the next section, it displays however some different interpretative features that justify its discussion in a separate publication [19].

3.2 Semiclassical quantization in finite volume

The application of the DHN method to the periodic kink (3.2) requires the solution of eq. (2.4) for the quantum fluctuations η_ω, which in this case takes the form

$$\left\{\frac{d^2}{d\bar{x}^2} + k^2\left(\bar{\omega}^2+1\right) - 2k^2\,\mathrm{sn}^2(\bar{x},k^2)\right\}\eta_{\bar{\omega}}(\bar{x}) \;=\; 0\;, \tag{3.7}$$

where $\mathrm{sn}(\bar{x},k^2)$ is the Jacobi elliptic function defined in Appendix B, and we have introduced the rescaled variables

$$\bar{x} \;=\; \frac{mx}{k}\;, \qquad \bar{\omega} \;=\; \frac{\omega}{m}\;. \tag{3.8}$$

Due to the periodic properties of $\phi_{cl}(x)$ expressed by eq. (3.2), the boundary condition (2.8) translates in the requirement for $\eta_{\bar{\omega}}(\bar{x})$

$$\eta_{\bar{\omega}}\left(\bar{x}+\frac{mR}{k}\right) \;=\; \eta_{\bar{\omega}}(\bar{x})\;\;. \tag{3.9}$$

Eq. (3.7) can be cast in the so–called Lamé form, which admits the two linearly independent solutions

$$\eta_{\pm a}(\bar{x}) \;=\; \frac{\sigma(\bar{x}+i\mathbf{K}'\pm a)}{\sigma(\bar{x}+i\mathbf{K}')}\,e^{\mp\bar{x}\,\zeta(a)}\;, \tag{3.10}$$

where the auxiliary parameter a is defined as a root of the equation

$$\mathcal{P}(a) \;=\; \frac{2-k^2}{3} - k^2\bar{\omega}^2\;\;. \tag{3.11}$$

The Weierstrass functions $\mathcal{P}(u)$, $\zeta(u)$ and $\sigma(u)$ are defined in Appendix C, where the Lamé equation and its relation with (3.7) are discussed in detail.

As it is usually the case for a Schrödinger–like equation with periodic potential, the spectrum of eq. (3.7) has a band structure, determined by the properties of the Floquet exponent

$$F(a) \;=\; 2i\left[\mathbf{K}\,\zeta(a) - a\,\zeta(\mathbf{K})\right]\;\;, \tag{3.12}$$

which is defined as the phase acquired by $\eta_{\pm a}$ in circling once the cylinder

$$\eta_{\pm a}(\bar{x}+2\mathbf{K}) \;=\; e^{\pm iF(a)}\,\eta_{\pm a}(\bar{x})\;\;.$$

We have two allowed bands for real $F(a)$, i.e.

$$0<\bar{\omega}^2<\frac{1}{k^2}-1 \quad\text{ and }\quad \bar{\omega}^2>\frac{1}{k^2}\;\;, \tag{3.13}$$

and two forbidden bands for $F(a)$ complex, i.e.

$$\bar{\omega}^2<0 \quad\text{ and }\quad \frac{1}{k^2}-1<\bar{\omega}^2<\frac{1}{k^2}\;\;. \tag{3.14}$$

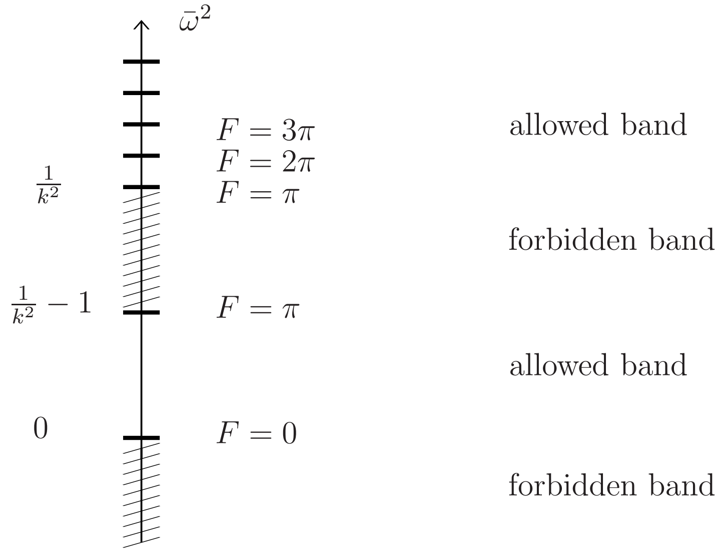

Figure 2: Spectrum of eq. (3.7)

The band $0 < \bar{\omega}^2 < \frac{1-k^2}{k^2}$ is described by $a = \mathbf{K} + iy$, where y varies between 0 and $\mathbf{K}'$ and, correspondingly, $F(a)$ goes from 0 to π. The other allowed band $\bar{\omega}^2 > \frac{1}{k^2}$ corresponds instead to $a = iy$ and, by varying y, $F(a)$ goes from π to infinity, as it is shown in Fig. 2.

By imposing the periodic boundary conditions (3.9) on the fluctuation $\eta(\bar{x})$, one selects the values of $\bar{\omega}^2$ for which the Floquet exponent is an even multiple of π, thus making the spectrum of eq. (3.7) discrete. These eigenvalues are $\bar{\omega}_0^2 = 0$, which is the zero mode associated with translational invariance and has multiplicity one, and the infinite series of points

$$\bar{\omega}_n^2 \equiv \frac{1}{k^2}\left[\frac{2-k^2}{3} - \mathcal{P}(iy_n)\right] \tag{3.15}$$

with multiplicity two, placed in the band $\bar{\omega}^2 > \frac{1}{k^2}$, with y_n determined by the equation

$$F(iy_n) = 2\mathbf{K}\, i\, \zeta(iy_n) + 2y_n\, \zeta(\mathbf{K}) = 2n\pi\,, \qquad n = 1, 2, \ldots \tag{3.16}$$

In the IR limit ($A \to 0$) the above spectrum goes to the one related to the standard background (2.9): the allowed band $0 < \bar{\omega}^2 < \frac{1}{k^2} - 1$, in fact, shrinks to the eigenvalue $\bar{\omega}_0^2 = 0$, while the other allowed band $\bar{\omega}^2 > \frac{1}{k^2}$ merges in the continuous part of the spectrum $\bar{\omega}_q^2 = 1 + q^2$.

It is useful to note that, although the R dependence of the frequencies (3.15) is quite implicit, since it passes through the inversion of eq. (3.3), nevertheless these are analytic functions of R and it is extremely simple to plot them. The corresponding curves, shown in Figure 3, provide an important piece of information, since they are nothing else but the energies of the excited states with respect to their ground state $\mathcal{E}_0(R)$.

To complete the analysis, it remains then to determine the finite volume ground state energy $\mathcal{E}_0(R)$ of the kink sector. In analogy with the infinite volume case (see eq. (2.11)),

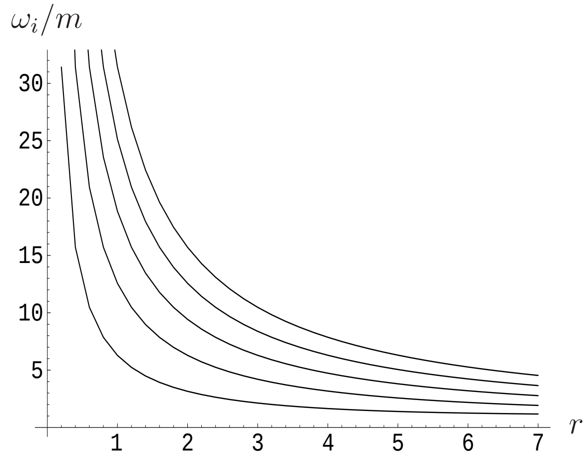

Figure 3: The first few levels defined in (3.15)

this is defined by

$$\mathcal{E}_0(R) \;=\; \mathcal{E}_{cl}(R) + \sum_{n=1}^{\infty} \omega_n(R) - \frac{\delta\mu^2}{\beta^2} \int\limits_{-R/2}^{R/2} dx \, [1 - \cos\beta\phi_{cl}] - \mathcal{E}_0^{\rm vac}(R) \;\; . \tag{3.17}$$

Before commenting in detail each of these terms, let's focus first on the main problem in deriving a closed expression for $\mathcal{E}_0(R)$, which consists in the evaluation of the infinite sum on the frequencies $\omega_n(R)$ or, better, in isolating its finite part. We need therefore a method for solving the transcendental equation (3.16) for $y_n(k^2)$ in order to make the expression (3.15) for the frequencies $\omega_n(k^2)$ explicit. As we have already seen for the classical energy, two kinds of expansion are possible, one in the elliptic modulus k and the other in the complementary modulus k', which are efficient approximation schemes in the small and large r regimes, respectively. Here for simplicity we only present the small r expansion. By taking into account the series expansion in k for $\mathbf{K}$, $\zeta(u)$ and $\mathcal{P}(u)$ (see Appendices B and C), we are led to look for a solution of eq. (3.16) in the form

$$y_n(k^2) \;=\; \sum_{s=0}^{\infty} (k^2)^s \, y_n^{(s)} \;\; . \tag{3.18}$$

Here we give the result for the first few coefficients $y_n^{(s)}$, $s = 0, 1, 2$:

$$\begin{aligned} y_n^{(0)} &= \text{arctanh}\frac{1}{2n} \;\; , \\ y_n^{(1)} &= \frac{1}{4}\, y_n^{(0)} \;\; , \\ y_n^{(2)} &= \frac{9}{64}\, y_n^{(0)} - \frac{n}{16(4n^2-1)^2} \;\; , \end{aligned} \tag{3.19}$$

which are those relevant in the analysis of the UV properties of the scaling function in Sect. 3.3. As a consequence, we obtain the following simple expression for the frequencies:

$$\frac{\omega_n}{m} = \frac{2n}{k}\left[1 - \frac{k^2}{4} - \frac{k^4}{64}\,\frac{20n^2-9}{4n^2-1} + O(k^6)\right] \,. \tag{3.20}$$

Comparing it order by order with the small–k expansion of eq. (3.3)

$$r = mR = \pi\, k\, \left[1 + \frac{k^2}{4} + \frac{9}{64}\,k^4 + O(k^6)\right] \,, \tag{3.21}$$

we finally obtain the explicit R–dependence

$$\frac{\omega_n(R)}{m} = \frac{2\pi}{r}\,n + \left(\frac{r}{2\pi}\right)^3 \frac{n}{4n^2-1} + \ldots \tag{3.22}$$

It is worth noting that this series expansion in r, which can be easily extended up to desired accuracy, efficiently approximates the exact energy levels also for rather large values of the scaling variable. Fig. 4 shows a numerical comparison between the first energy level given by (3.15) and its approximate expression (3.22), and for the higher levels it is possible to see that the agreement is even better.

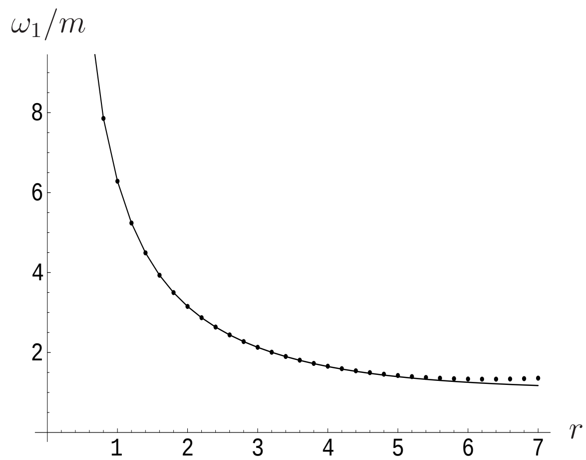

Figure 4: Comparison between the exact energy level ω_1/m given by (3.15) (continuous line) and the approximate expression (3.22) (dotted line).

With the above analysis, the sum over frequencies in the ground state energy (3.17) takes the form

$$\sum_{n=1}^{\infty} \frac{\omega_n(R)}{m} = \frac{2\pi}{r}\sum_{n=1}^{\infty} n + \left(\frac{r}{2\pi}\right)^3 \sum_{n=1}^{\infty} \frac{n}{4n^2-1} + \ldots \tag{3.23}$$

As we will see below, the subtraction of counterterm and vacuum energy in (3.17) leads to the cancellation of all the divergencies, producing a finite expression for the ground state energy in the kink sector.

Moving now to the analysis of the remaining terms in (3.17), a similar series expansion can be easily performed on each of them. The classical energy $\mathcal{E}_{cl}(R)$, given in eq. (3.4), has already be treated in this way in eq. (3.6). The finite volume counterterm (C.T.), where the one–loop mass renormalisation is given by

$$\delta\mu^2 \;=\; -\frac{m^2\beta^2}{8\pi}\,\frac{2\pi}{R}\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{m^2+\frac{(2n\pi)^2}{R^2}}} \tag{3.24}$$

and ϕ_{cl} is given by (3.2), takes the explicit form

$$\text{C.T.} \;=\; m\left[k\mathbf{K}\left(k^2\right)-\frac{\mathbf{K}\left(k^2\right)-\mathbf{E}\left(k^2\right)}{k}\right]\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{(2n\pi)^2+r^2}}\quad . \tag{3.25}$$

The first terms of its expansion in R are then

$$\frac{\text{C.T.}}{m} \;=\; \frac{1}{4}+\frac{r}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\;-\;\frac{r^2}{32\pi^2}-\frac{1}{4}\left(\frac{r}{2\pi}\right)^3\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{1}{n^3}\right)+\ldots\;, \tag{3.26}$$

Finally, the vacuum energy $\mathcal{E}_0^{\text{vac}}(R)$ is the one precisely computed in Sec. 2.3. Since its role is to cancel certain divergencies present in the other terms of $\mathcal{E}_0(R)$, in complete analogy with the infinite volume case (see eq. (2.11)), we will now consider its "naive" formulation, given by

$$\frac{\mathcal{E}_0^{\text{vac}}(R)}{m} \;=\; \frac{1}{2m}\sum_{n=-\infty}^{\infty}\sqrt{\left(\frac{2n\pi}{R}\right)^2+m^2} \;=\; \frac{1}{2}+\frac{2\pi}{r}\sum_{n=1}^{\infty}n+\frac{r}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{8}\left(\frac{r}{2\pi}\right)^3\sum_{n=1}^{\infty}\frac{1}{n^3}+\ldots \tag{3.27}$$

Hence, in the final expression for the ground state energy all the divergent series present in the sum over frequencies, in the counterterm and in the vacuum energy cancel out, and one obtains

$$\frac{\mathcal{E}_0(R)}{m} \;=\; \frac{2\pi}{r}\,\frac{\pi}{\beta^2}-\frac{1}{4}+\frac{1}{\beta^2}\,r-\frac{1}{8}\left(\frac{r}{2\pi}\right)^2-\left(\frac{r}{2\pi}\right)^3\left[\frac{1}{8}\,\zeta(3)-\frac{1}{4}(2\,\log 2-1)-\frac{\pi}{2\beta^2}\right]+\ldots\;, \tag{3.28}$$

where we have used [30]

$$\sum_{n=1}^{\infty}\frac{2n^2-1}{8n^3(4n^2-1)} \;=\; \frac{1}{8}\,\zeta(3)-\frac{1}{4}(2\,\log 2-1)$$

in order to evaluate explicitly the coefficient of the r^3 term.

Repeating the above calculations, one can also easily write the finite expressions of the excited energy levels (2.5), whose series expansion in r is given by

$$\begin{aligned}\frac{\mathcal{E}_{\{k_n\}}(R)}{m} \;=\;& \frac{2\pi}{r}\left(\frac{\pi}{\beta^2}+\sum_n k_n\,n\right)-\frac{1}{4}+\frac{1}{\beta^2}\,r-\frac{1}{8}\left(\frac{r}{2\pi}\right)^2+ \\ & -\left(\frac{r}{2\pi}\right)^3\left[\frac{1}{8}\zeta(3)-\frac{1}{4}(2\,\log 2-1)-\frac{\pi}{2\beta^2}+\sum_n k_n\,\frac{n}{4n^2-1}\right]+\ldots\end{aligned} \tag{3.29}$$

where $\{k_n\}$ is a set of integers defining a particular excited state of the kink.

3.3 UV–IR correspondence

The semiclassical quantization of the periodic kink (3.2) provides us with analytic expressions, albeit implicit, for the scaling functions in the kink sector for arbitrary values of the scale $r = mR$. These quantities control analytically the interpolation between the Hilbert spaces of the ultraviolet (UV) and infrared (IR) limiting theories. It is worth to note that, although we obtain them in the framework of a particle–like description proper of the IR limit, the kink background (3.2) is intrinsically formulated on a finite size, leading to the possibility of extracting UV data. Hence, it is important to check whether our scaling functions reproduce both the expected results for the IR ($r \to \infty$) and UV ($r \to 0$) limits.

Concerning the IR behaviour, we have seen in the previous Sections that in the $R \to \infty$ limit all the quantities in exam, i.e. the classical solution, its classical energy and the stability frequencies, correctly reach their asymptotic values. In addition, it is also possible to perform a simple and interesting analysis of the first correction to the kink mass for large R. According to Lüscher's analysis [24], the mass of a particle in a large but finite volume has to approach exponentially its asymptotic value, in a way controlled by the scattering data of the infinite volume theory. Restricting for simplicity our analysis to the leading term in β in the kink mass (which, in our approch, is simply given by the classical energy), we have then to compare the expansion presented in (3.5) with the term that dominates Lüscher's formula for small β. This is given by [24, 25]

$$M(R) - M_\infty \;=\; -m_b \sin u_{kb}^k \, R_{kbk} \; e^{-m_b \sin u_{kb}^k R} + \cdots \;, \tag{3.30}$$

where R_{kbk} is the residue (multiplied by $-i$) of the kink–breather S-matrix on the pole at $\theta = i u_{kb}^k$, i.e.

$$R_{kbk} \;=\; -i \mathrm{Res}\, S_{kb}(\theta = i u_{kb}^k) \;\; . \tag{3.31}$$

Using the kink–breather S–matrix [3]

$$S_{kb}(\theta) \;=\; \frac{\sinh\theta + i\cos\frac{\gamma}{16}}{\sinh\theta - i\cos\frac{\gamma}{16}} \quad , \quad \gamma \;=\; \frac{\beta^2}{1-\beta^2/8\pi} \tag{3.32}$$

and selecting its s-channel pole $\theta^* = i u_{kb}^k = i\left(\frac{\pi}{2} + \frac{\gamma}{16}\right)$, we find

$$R_{kbk} \;=\; -2\,\mathrm{cotg}\frac{\gamma}{16} \;\; . \tag{3.33}$$

Substituting in (3.30), for small β^2 we have

$$M(R) - M_\infty \;=\; m\,\frac{32}{\beta^2}\, e^{-r} + \cdots \;, \tag{3.34}$$

which therefore reproduces eq. (3.5). It is a remarkable fact that the classical energy alone, being the leading term in the mass for $\beta^2 \to 0$, contains the IR scattering information which controls the large–distance behaviour of $\mathcal{E}_0(R)$.

The UV behaviour for $r \to 0$ of the ground state energy $E_0(R)$ of a given off–critical theory is known to be related instead to the Conformal Field Theory (CFT) data $(\Delta, \bar{\Delta}, c)$ of the corresponding critical theory and to the bulk energy term as [2]

$$E_0(R) \simeq \frac{2\pi}{R}\left(\Delta + \bar{\Delta} - \frac{c}{12}\right) + BR + \cdots \tag{3.35}$$

where c is the central charge, $\Delta + \bar{\Delta}$ is the lowest anomalous dimension in a given sector of the theory and B the bulk coefficient.

For the Sine–Gordon model the bulk energy term is given by [8, 26]

$$B = 16\,\frac{m^2}{\gamma^2}\,\tan\frac{\gamma}{16}\,, \tag{3.36}$$

while its UV limit is described by the CFT given by the gaussian action with $c = 1$

$$\mathcal{A}_{\rm CFT} = \frac{1}{2}\,g \int d^2x\, \partial_\mu\phi\, \partial^\mu\phi\,, \tag{3.37}$$

where the free bosonic field is compactified on a circle of radius $\mathcal{R}$. The various sectors of this CFT are labelled by two integers, s and n: s is the momentum index, while n is the winding number, related to the boundary condition imposed on ϕ

$$\phi(x + R, t) = \phi(x, t) + 2\pi n\,\mathcal{R}\,. \tag{3.38}$$

Let $\mid s, n\rangle$ be the states carrying the lowest anomalous dimension in each sector. They are created by the vertex operators [27]

$$V_{s,n}(z, \bar{z}) =: \exp\left[i\alpha^+_{s,n}\varphi(z) + i\alpha^-_{s,n}\bar{\varphi}(\bar{z})\right]\, :\,,$$

i.e.

$$\mid s, n\rangle = V_{s,n}(0, 0)\ \mid {\rm vac}\rangle\ , \tag{3.39}$$

where

$$\begin{aligned} \alpha^{\pm}_{s,n} &= \frac{s}{\mathcal{R}} \pm 2\pi g\, n\mathcal{R}\ \ ; \\ \phi(x,t) &= \varphi(z) + \bar{\varphi}(\bar{z})\ \ . \end{aligned}$$

Their conformal dimensions are given by

$$\Delta_{s,n} = 2\pi g\left(\frac{s}{4\pi g\,\mathcal{R}} + \frac{1}{2}\,n\mathcal{R}\right)^2\,, \qquad \bar{\Delta}_{s,n} = 2\pi g\left(\frac{s}{4\pi g\,\mathcal{R}} - \frac{1}{2}\,n\mathcal{R}\right)^2\ . \tag{3.40}$$

The vacuum sector is described by $s = n = 0$, with $\Delta_{vac} + \bar{\Delta}_{vac} = 0$. If we now define $\mathcal{R} = \frac{1}{\sqrt{g}\,\beta}$ and fix the normalization constant to the value[3] $g = 1$, then the kink sector in SG, defined by the boundary condition (2.8), naturally corresponds to the sector characterized by $s = 0, n = 1$, in which the lowest anomalous dimension is

$$\Delta_{0,1} + \bar{\Delta}_{0,1} \;=\; \frac{\pi}{\beta^2} \;\;. \tag{3.41}$$

The conformal vertex operator $V_{0,1}$ has been put in exact correspondence with the soliton–creating operator of SG in Ref. [28].

The question to be addressed now is whether the small r expansion of $\mathcal{E}_0^{vac}(R)$ and $\mathcal{E}_0(R)$ given by eqs. (2.26) and (3.28) reproduces, in semiclassical approximation, the above data controlling the UV limit of SG model.

For the vacuum sector, comparing (2.26) with (3.35), we correctly obtain $c = 1$ and $\Delta_{vac} = \bar{\Delta}_{vac} = 0$. We do not expect, however, to obtain the bulk term B relative to SG model by looking at (2.26), simply because the semiclassical expression of the ground state energy in the vacuum sector applies equally well to any theory which has a quadratic expansion near the vacuum state. Namely, apart from the value of the mass m, eq. (2.26) is a universal expression that does not refer then to SG model.

The kink scaling function (3.28) has instead a richer structure. The obtained scaling dimension

$$\Delta + \bar{\Delta} \;=\; \frac{\pi}{\beta^2} \tag{3.42}$$

is the expected one for the soliton-creating operator in Sine-Gordon while the central charge contribution $c = 1$ is absent, simply because in (3.28) we have subtracted the vacuum ground state energy from the kink one[4]. Moreover, the bulk coefficient $B = \frac{m^2}{\beta^2}$ present in (3.28) correctly reproduces the semiclassical limit of the exact one, given in eq. (3.36). In principle, this bulk term should be present in all the energy levels, included the ground state energy in the vacuum sector, but its non–perturbative nature makes impossible to see it in the semiclassical expansion around the vacuum solution, which is in fact purely perturbative. Hence it is not surprising that to extract the bulk energy term we have to look at the kink ground state energy, in virtue of the non–perturbative nature of the corresponding classical solution.

Finally, the expression (3.29) for the excited energy levels explicitly show their correspondence with the conformal descendants of the kink ground state. In fact, their anomalous dimension is given by

$$\Delta_{\{k_n\}} + \bar{\Delta}_{\{k_n\}} = \frac{\pi}{\beta^2} + \sum_n k_n\, n \;. \tag{3.43}$$

[3] Note that the usual normalization adopted in the CFT literature is instead $g = \frac{1}{4\pi}$.

[4] The value $c = 1$, coming out from the regularization of the leading term of the series on the frequencies (3.22), is in fact exactly cancelled by the same term in the vacuum energy.

The successful check with known UV and IR asymptotic behaviours confirms the ability of the semiclassical results to describe analytically the scaling functions of SG model in the one–kink sector. It would be interesting to further test them at arbitrary values of r through a numerical comparison with the results of [8, 9] in an appropriate range of parameters. This was not pursued here because the results presently available in the literature were obtained for values of β which are beyond the semiclassical regime and moreover the energy levels were plotted as functions of a different scaling variable, i.e. the one defined in terms of the kink mass. We hope however to come back to this problem in the future.

4 Form factors and correlation functions

The semiclassical scaling functions, derived in Sect. 3, provide an important information about the finite size effects in SG model. As in the infinite volume case, however, the complete description of the finite volume QFT requires to find, in addition to the energy eigenvalues (3.29), the kink form factors and the correlation functions of local operators. This section is devoted to the analysis of this problem, i.e. to the determination of the finite volume form factors and the corresponding spectral functions.

4.1 Infinite volume form factors

It is useful to initially recall some basic definitions and results concerning semiclassical form factors for the SG model in infinite volume. As mentioned in Sect. 2, the relation between kink solutions and form factors was established by Goldstone and Jackiw [13], who showed that the matrix element of the field ϕ between two asymptotic one–kink states is given, at leading order in the semiclassical regime, by the Fourier transform of the classical solution describing the kink itself (see also [14] for further developments). This remarkable result, however, had the drawback of being formulated non–covariantly in terms of the kink space–momenta. It was refined in [18] with a covariant formulation in terms of the rapidity variable θ of the kink, defined in terms of its energy and momentum as $E = M\cosh\theta\,,\ p = M\sinh\theta$. In the semiclassical regime, there are moreover further simplifications: in fact, the mass of the kink can be approximated by its classical energy $M \simeq M_{cl} = \frac{8m}{\beta^2}$ whereas its rapidity can be assumed to be very small, i.e. $\theta = O(\beta^2)$, thus obtaining $E \simeq M_{cl}\,,\ p \simeq M_{cl}\theta$. Hence, the refined form of Goldstone and Jackiw result is given by

$$\langle\theta_1 \mid \phi(0) \mid \theta_2\rangle \;=\; M_{cl} \int_{-\infty}^{\infty} da\, e^{-iM_{cl}(\theta_1-\theta_2)a}\, \phi_{cl}(a) \;\;, \tag{4.1}$$

where $|\,\theta_i\rangle$ are asymptotic one–kink states. Moreover, it is also possible to prove that the form factor of an operator expressible as a function of ϕ is given by the Fourier transform of the same function of ϕ_{cl}. For instance, the form factor of the energy density operator ε can be computed performing the Fourier transform of $\varepsilon_{cl}(x) = \frac{1}{2}\left(\frac{d\phi_{cl}}{dx}\right)^2 + V[\phi_{cl}]$. With this covariant formulation, the matrix element (4.1) can be continued to the crossed channel, and from its pole structure one can easily extract the spectrum of the bound states of the theory, even in non–integrable cases, as discussed in [17, 18].

In [18] we have checked that the form factor (4.1) obtained from the infinite volume kink (2.9) reproduces the semiclassical limit of the exact one, derived in [29]. Here we would like to present a more quantitative comparison which permits to conclude that formula (4.1), though proven under the semiclassical assumption of small coupling and small rapidities, remarkably extends its validity to finite values of the coupling and to a large range of the rapidities. Consider, for instance, the form factor of the energy operator (up to a normalization N) $F(\theta) = N\langle\,\theta_2\,|\,\epsilon(0)\,|\,\theta_1\,\rangle$, whose semiclassical and exact expressions are given, respectively, by

$$F_{\text{semicl.}}(\theta) = \frac{\theta}{2}\,\frac{1}{\sinh\left[\frac{4\pi}{\beta^2}\,\theta\right]} \tag{4.2}$$

$$F_{\text{exact}}(\theta) = \sinh\frac{\theta}{2}\,\frac{1}{\sinh\frac{\theta}{2\xi}}\;G(\theta) \tag{4.3}$$

where $\xi = \gamma/8\pi$ and

$$G(\theta) = \exp\left[\int_0^\infty \frac{dt}{t}\,\frac{\sinh\frac{t}{2}(1-\xi)}{\sinh\frac{t}{2}\xi\;\;\cosh\frac{t}{2}}\,\frac{\sin^2\frac{\theta\,t}{2\pi}}{\sinh t}\right]\;. \tag{4.4}$$

Fig. 5 shows how, for small values of the coupling, the agreement between the two functions is very precise for the whole range of the rapidity. Furthermore, the discrepancy between exact and semiclassical formulas at larger values of β can be simply cured, in our example, by substituting the bare coupling β^2 with its renormalized expression γ into the semiclassical result (4.2), as shown in Fig. 6. Hence we can conclude that the monodromy factor (4.4), which is the relevant quantity missing in our approximation, does not play a significant role in the quantitative evaluation of the form factor even for certain finite values of the coupling[5].

As we have already mentioned, the exactness (or very high accuracy, as in this case) of the semiclassical results is a peculiar feature of SG model in infinite volume, obtained

[5]It is easy to understand the reason of this conclusion in the above example: at small values of θ we have $G(\theta) \simeq 1$, whereas for $\theta \to \infty$, when $G(\theta)$ may contribute, the whole form factor goes anyway to zero. Similar conclusion can be reached for all other form factors which vanish at $\theta \to \infty$.

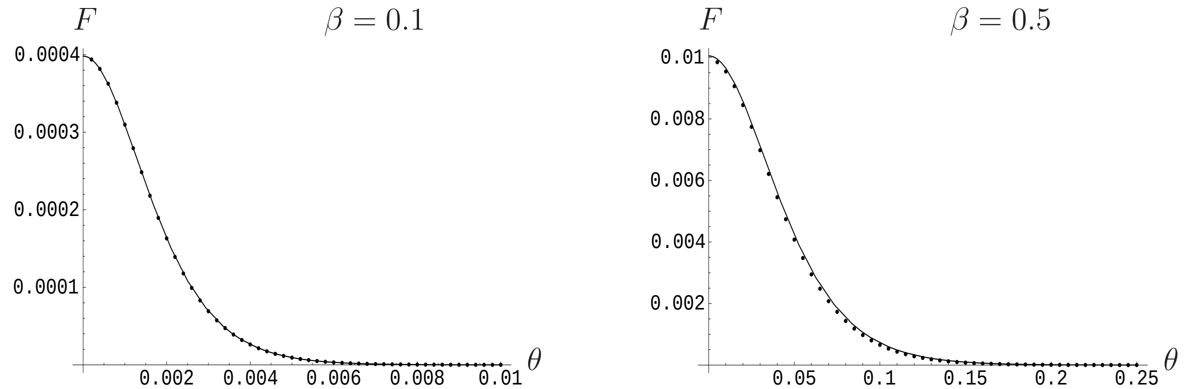

Figure 5: Comparison between the exact function F given by (4.3) (continuous line) and its semiclassical approximation (4.2) (dotted line), at $\beta = 0.1$ and $\beta = 0.5$.

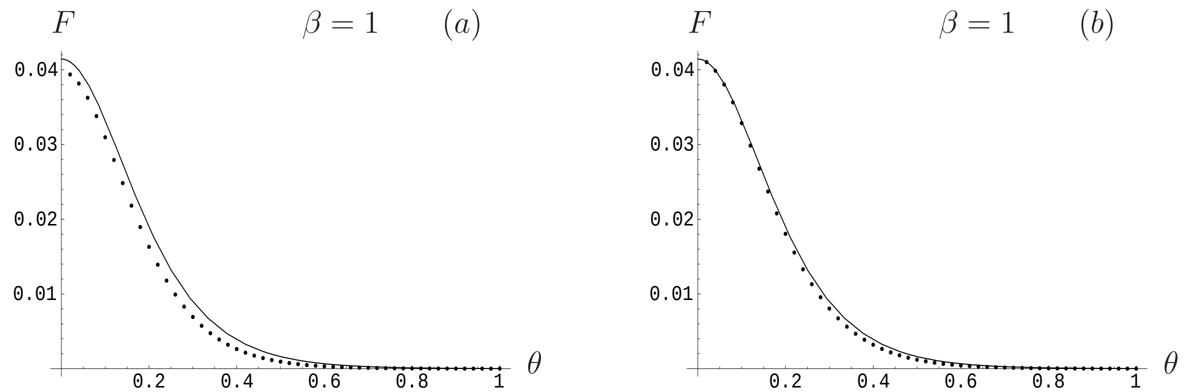

Figure 6: Comparison, at $\beta = 1$, between (a) the exact function F given by (4.3) (continuous line) and its semiclassical approximation (4.2) (dotted line), (b) the exact function F given by (4.3) (continuous line) and its semiclassical approximation (4.2) with the substitution $\beta^2 \to \gamma$ (dotted line).

with the "dressing" $\beta^2 \to \gamma$. An interesting problem to be studied is whether similar phenomena take place for the semiclassical scaling functions and form factors in finite volume as well. An indication on this issue could be found by extending to finite volume the analysis of higher loop quantum corrections which, in the semiclassical approach, are obtained by keeping cubic (and higher) powers of η in the expansion of $V(\phi_{\rm cl} + \eta)$ [14].

4.2 Semiclassical spectral functions on the cylinder

The generalization of the above construction to the case of finite volume has been proposed in [18], where we have shown how to estimate the leading semiclassical behaviour of the spectral function on the cylinder under the same hypotheses of the infinite volume case.

In fact, the matrix element

$$f(\theta_n) \;=\; \langle p_{n_2}|\, \phi(0)\, |p_{n_1}\rangle \;, \tag{4.5}$$

of the basic field ϕ between two kink eigenstates of the finite volume hamiltonian can be expressed, at leading order, as the Fourier transform of the corresponding classical solution:

$$f(\theta_n) \;\equiv\; M(R) \int_{-R/2}^{R/2} da\, e^{i\,M(R)\theta_n a}\, \phi_{cl}(a) \;, \tag{4.6}$$

$$\phi_{cl}(a) \;\equiv\; \frac{1}{R\,M(R)} \sum_{n=-\infty}^{\infty} e^{-i\,M(R)\theta_n a}\, f(\theta_n) \;. \tag{4.7}$$

Here we have denoted the states by the so-called "quasi-momentum" variable p_n, which corresponds to the eigenvalues of the translation operator on the cylinder (even multiples of π/R), and we have defined θ_n as the "quasi-rapidity" of the kink states

$$\frac{2n\pi}{R} \;=\; p_n \;=\; M(R)\,\sinh\theta_n \;\simeq\; M(R)\,\theta_n \;, \tag{4.8}$$

where $M(R)$ is the classical mass of the finite–volume kink. As shown in [18], the crossed channel form factor can be obtained at leading order via the change of variable $\theta \to i\pi - \theta$:

$$F_2(\theta_n) \;=\; \langle\, 0\,|\, \phi(0)\, |\bar{p}_{n_2}\, p_{n_1}\rangle \;=\; f(i\pi - \theta_n) \;, \tag{4.9}$$

and the leading terms in the spectral function on the cylinder are given by

$$\hat{\rho}(E_k, p_k) \;=\; 2\pi\delta(E_k)\delta(p_k)|<0|\,\phi(0)|\,0>|^{\,2} + \frac{\pi}{4}\,\frac{\delta\left(\frac{E_k}{M} - 2\right)}{M^2}\,\left|F_2\left(i\pi - \frac{p_k}{M}\right)\right|^2 \;. \tag{4.10}$$

The procedure described above has been introduced in [18] for the construction of form factors for the kink backgrounds in the SG model and the broken ϕ^4 field theory, both defined on a cylindrical geometry with *antiperiodic* boundary conditions. In what follows we will apply it instead to the case of SG model with *periodic* boundary conditions. The corresponding finite volume form factor (4.6) can be written in terms of the soliton background (3.2):

$$f(\theta_n) = M \int_{-R/2}^{R/2} da\, e^{i\,M\theta_n a}\left[\frac{\pi}{\beta} + \frac{2}{\beta}\,\mathrm{am}\left(\frac{mx}{k}, k^2\right)\right] = \tag{4.11}$$

$$= \frac{2\pi}{\beta}\left\{\frac{M}{2}\,R\,\delta_{M\theta_n,0} - i\,\frac{1-\delta_{M\theta_n,0}}{\theta_n}\left[\cos\left(M\theta_n\,R/2\right) - \frac{\sin\left(M\theta_n\,R/2\right)}{M\theta_n R/2}\right] + i\,\frac{1}{\theta_n\,\cosh\left(k\,\mathbf{K}'\frac{M}{m}\,\theta_n\right)}\right\}\;.$$

In order to obtain this result one has to compare the inverse Fourier transform (4.7) with the expansion [30]

$$\mathrm{am}(u) \;=\; \frac{\pi\, u}{2\mathbf{K}} + \sum_{n=1}^{\infty} \frac{1}{n\,\cosh\left[n\pi\frac{\mathbf{K}'}{\mathbf{K}}\right]}\;\sin\left[n\pi\frac{u}{\mathbf{K}}\right]\;. \tag{4.12}$$

The form factor (4.11) has the correct IR limit[6], and leads to the following expressions for $F_2(\theta)$ and for the spectral function[7]:

$$\begin{aligned} F_2(\theta_n) \;&=\; \frac{4\pi i}{\beta\,\hat{\theta}_n}\left\{\frac{1}{\cosh\left[k\,\mathbf{K}'\,\frac{M}{m}\,\hat{\theta}_n\right]}+\right.\\ &\left.-\left(1-\delta_{\hat{\theta}_n,0}\right)\left[\cos\left(M\,\hat{\theta}_n\,R/2\right)-\frac{\sin\left(M\hat{\theta}_n\,R/2\right)}{M\,\hat{\theta}_n\,R/2}\right]\right\} \end{aligned} \tag{4.13}$$

$$\begin{aligned} \hat{\rho}(E_n,p_n) \;&=\; 4\pi^3\,\delta\left(\frac{E_n}{M}-2\right)\frac{1}{\beta^2(p_n)^2}\left\{\frac{1}{\cosh\left[\frac{k\,\mathbf{K}'}{m}\,p_n\right]}+\right.\\ &\left.-\left(1-\delta_{p_n,0}\right)\left[\cos\left(p_n\,R/2\right)-\frac{\sin\left(p_n\,R/2\right)}{p_nR/2}\right]\right\}^2\;, \end{aligned} \tag{4.14}$$

where $\hat{\theta} \,=\, i\pi - \theta$. Note that the finite volume dependence of both the form factor (4.13) and the spectral function (4.14) is not restricted to the second term only. The $k\,\mathbf{K}'(k^2)\,M(R)$ factor in the first term carries the main R-dependence, although it is not manifest but implicitly defined by eq. (3.3).

Another quantity of interest is the two–point function $\langle\, 0 \mid \varepsilon(x)\varepsilon(0) \mid 0\,\rangle$ of the energy density operator. One can calculate it by evaluating its spectral function

$$\rho_\varepsilon(p^2) = \int_{-R/2}^{R/2} dx\,\langle\, 0 \mid \varepsilon(x)\varepsilon(0) \mid 0\,\rangle\, e^{-ip\cdot x} \tag{4.15}$$

in terms of the form factors of $\varepsilon(x)$, similarly to what we have done for the SG field ϕ above. In order to find the semiclassical form factor

$$f_\varepsilon(\theta_n) \;=\; \langle p_{n_2}|\,\varepsilon(0)\,|p_{n_1}\rangle\;, \tag{4.16}$$

we need to compute the Fourier transform of

$$\varepsilon(\phi_{\rm cl}) \;=\; \frac{2m^2}{\beta^2k^2}(1+k^2) - \frac{4m^2}{\beta^2}\,\mathrm{sn}^2\left(\frac{mx}{k}\right)\;. \tag{4.17}$$

[6]The functions $\frac{\cos(xR/2)}{x}$ and $\frac{\sin(xR/2)}{x^2R/2}$ can be shown to tend to zero in the distributional sense for $R\to\infty$.

[7]Here we are considering the matrix elements on the antisymmetric combinations of kink and antikink.

This can be easily obtained from the following expansion

$$\mathrm{sn}^2 u \;=\; \frac{1}{k^2 \mathbf{K}} \left\{ \mathbf{K} - \mathbf{E} - \frac{\pi^2}{\mathbf{K}} \sum_{n=1}^{\infty} \frac{n\,\cos\frac{n\pi u}{\mathbf{K}}}{\sinh\frac{n\pi \mathbf{K}'}{\mathbf{K}}} \right\} \;, \tag{4.18}$$

and we finally have

$$f_\varepsilon(\theta_n) \;=\; M^2 \left\{ \delta_{M\theta_n,0} \,+\, \frac{4\pi}{\beta^2}\, \frac{\theta_n}{\sinh\left(k\,\mathbf{K}' \frac{M}{m}\,\theta_n\right)} \right\} \;. \tag{4.19}$$

The corresponding semiclassical spectral function is thus given by

$$\hat{\rho}_\varepsilon(E_n, p_n) = \frac{4\pi^3}{\beta^4}\,\delta\left(\frac{E_n}{M} - 2\right) \frac{p_n^2}{\sinh^2\left(\frac{k\,\mathbf{K}'}{m}\,p_n\right)} \;. \tag{4.20}$$

It is worth to mention that it is also possible to obtain the two–point functions of certain vertex operators $V_b^{\pm}(x,t) = e^{\pm i\beta b\phi(x,t)}$ (for $b = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$), since the required Fourier expansion formulas of the type (4.18) are known in these cases [31].

5 Further directions

In this paper we have shown how the semiclassical methods can provide an analytic description of finite size effects in two–dimensional quantum field theories displaying degenerate vacua. In particular, we have applied these techniques to study the SG model, quantizing its kink solution on the cylinder. The scaling functions of the ground (and excited) states, as well as the form factors and two–point functions of different operators, allowed us to build the one–kink sector (i.e. $Q_{\mathrm{top}} = \pm 1$) of the corresponding Hilbert space of states.

The next step in this program is the extension of the DHN method to describe the multi–kink states ($Q_{\mathrm{top}} = \pm 2, \pm 3, \ldots$) as well as the non–vacua ("breather"–like) part of the $Q_{\mathrm{top}} = 0$ sector. These states are related to certain time–dependent solutions of SG model on the cylinder, i.e. to the finite volume analog of soliton–soliton, soliton–antisoliton and breather solutions. Although more complicated from the technical point of view, the determination of these classical solutions and the study of their scaling functions and form factors is a well stated open problem in the semiclassical framework, which deserves further attention.

One of the advantages of the semiclassical method is that it works equally well for both integrable and non–integrable models, if they admit kink–type solutions. In fact, we have chosen to test the efficiency of the semiclassical quantization on the example of SG model, mainly because it leads to the simplest $N = 1$ Lamé equation. Static elliptic

solutions for other models can be easily obtained by integrating equation (2.2) with $A \neq 0$ and appropriate boundary conditions. This was done, for instance in [18], where we have derived the form factors between kink states in the broken ϕ^4 model on the cylinder with twisted boundary conditions. In this case, the quantization of the finite volume kink involves a Lamé equation with $N = 2$. Lamé equations with $N > 2$ are also expected to enter the quantization of other theories.

Finally, the semiclassical method seems to be suited also for the description of finite geometries with boundaries, say with Dirichlet or Neumann boundary conditions. Due to the physical significance of this kind of systems, an interesting problem is the semiclassical computation of the relative energy levels, a subject that will be discussed in a forthcoming publication [19].

Acknowledgements

This work was partially supported by the Italian COFIN contract "Teoria dei Campi, Meccanica Statistica e Sistemi Elettronici" and by the European Commission TMR programme HPRN-CT-2002-00325 (EUCLID). G.S. thanks FAPESP for the financial support. G.M would like to thank the Instituto de Fisica Teorica in San Paulo for the warm hospitality.

A Free theory quantization on a finite geometry

Let us consider a free bosonic field $\phi(x,t)$ of mass m defined on a cylinder of circumference R, i.e. satisfying the periodic boundary condition

$$\phi(x+R,t) \;=\; \phi(x,t) \;\; . \tag{A.1}$$

Imposing the equation of motion and the commutation relation

$$[\,\phi(x,t),\Pi(y,t)\,] \;=\; i\delta_P(x-y) \;\; , \tag{A.2}$$

where $\Pi(x,t)=\frac{\partial\phi}{\partial t}(x,t)$ is the conjugate momentum of the field whereas

$$\delta_P(x) \;=\; \frac{1}{R}\sum_{n=-\infty}^{\infty} e^{\frac{2\pi i n}{R}\,x} \quad , \quad \delta_P(x+R) \;=\; \delta_P(x)$$

is the periodic version of the Dirac delta function, we obtain the mode expansion of the field $\phi(x,t)$. This is given by

$$\phi(x,t) \;=\; \sum_{n=-\infty}^{\infty}\frac{1}{2\omega_n R}\left[\,A_n\,e^{i(p_n x-\omega_n t)}+A^{\dagger}\,e^{-i(p_n x-\omega_n t)}\,\right] \;\; , \tag{A.3}$$

where

$$[A_n, A_m^{\dagger}] \;=\; \delta_{n,m} \;\; ,$$

and

$$\omega_n \;=\; \sqrt{p_n^2+m^2} \quad , \quad p_n \;=\; \frac{2\pi n}{R} \quad n=0,\pm 1,\ldots \tag{A.4}$$

Using the above expansion together with the commutation relation of A and $A^{\dagger}$, it is easy to compute the propagator of the field, given by

$$\Delta_F(x-x',t-t') \;=\; \langle\phi(x,t)\phi(x',t')\rangle \;=\; \sum_{n=-\infty}^{\infty}\frac{1}{2\omega_n R}\,e^{-i[\omega_n(t-t')-p_n(x-x')]} \;\; . \tag{A.5}$$

The vacuum expectation value of the operator $\phi^2(x,t)$ is then formally given by

$$\langle\phi^2(x,t)\rangle \;=\; \Delta_F(0) \tag{A.6}$$

and, by translation invariance, is independent from x and t. However this expression is divergent and needs therefore to be regularized. Analogously to what has been done in the text for the ground state energy $\mathcal{E}_0^{\rm vac}(R)$, we need to subtract the corresponding expression in the infinite volume, so that the finite quantity, simply denoted by $\phi_0^2(R)$, satisfies the usual normalization condition

$$\lim_{R\to\infty}\phi_0^2(R) \;=\; 0 \;\; .$$

Hence we define

$$\phi_0^2(R) \;=\; \frac{1}{2R} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{\left(\frac{2\pi n}{R}\right)^2 + m^2}} \;-\; \frac{1}{2R} \int\limits_{-\infty}^{\infty} dn \frac{1}{\sqrt{\left(\frac{2\pi n}{R}\right)^2 + m^2}} \;. \tag{A.7}$$

Isolating its zero mode, the series needs just one subtraction, i.e.

$$\mathcal{S}(r) \;\equiv\; \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2}} \;=\; \sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2}} - \frac{1}{n} \right\} + \sum_{n=1}^{\infty} \frac{1}{n} \;. \tag{A.8}$$

$(r = mR)$. In the above expression, the first series is now convergent whereas the second series, which is divergent, has to be combined with a divergence coming from the integral. Indeed we have

$$\mathcal{I}(r) \;\equiv\; \int\limits_0^{\infty} dn \frac{1}{\sqrt{n^2 + \left(\frac{r}{2\pi}\right)^2}} \;=\; \lim_{\Lambda\to\infty} \left\{ \ln 2\Lambda - \ln \frac{r}{2\pi} \right\} - \lim_{\Lambda\to\infty} \ln\Lambda + \lim_{\Lambda\to\infty} \ln\Lambda \;, \tag{A.9}$$

and the last term can be used to compose (2.25). Collecting the above expressions, it is now easy to see that $\phi_0^2(R)$ coincides with the one obtained doing the calculation in the other channel, i.e. at a finite temperature. In fact, using the results of Ref. [21], this quantity can be expressed as

$$\phi_0^2(R) \;=\; \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{1}{e^{r\cosh\theta} - 1} \;\;, \tag{A.10}$$

whose expansion in r is given by

$$\phi_0^2(R) \;=\; \frac{1}{2r} + \frac{1}{2\pi}\left(\log\frac{r}{2\pi} + \gamma_E - \log 2\right) + \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{(2n\pi)^2 + r^2}} - \frac{1}{2n\pi} \right) \;. \tag{A.11}$$

Also this result could have been directly obtained computing only the finite part of the integral and using the prescription (2.30).

B Elliptic integrals and Jacobi's elliptic functions

In this appendix we collect the definitions and basic properties of the elliptic integrals and functions used in the text. Exhaustive details can be found in [30].

The complete elliptic integrals of the first and second kind, respectively, are defined as

$$\mathbf{K}(k^2) \;=\; \int\limits_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2\alpha}} \;, \qquad \mathbf{E}(k^2) \;=\; \int\limits_0^{\pi/2} d\alpha \sqrt{1 - k^2\sin^2\alpha} \;. \tag{B.1}$$

The parameter k, called elliptic modulus, has to be bounded by $k^2 < 1$. It turns out that the elliptic integrals are nothing but specific hypergeometric functions, which can be easily expanded for small k:

$$\begin{aligned}\mathbf{K}(k^2) &= \frac{\pi}{2}\, F\left(\frac{1}{2},\frac{1}{2},1;k^2\right) = \frac{\pi}{2}\left\{1+\frac{1}{4}\,k^2+\frac{9}{64}\,k^4+\ldots+\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}+\ldots\right\}, \quad \text{(B.2)}\\ \mathbf{E}(k^2) &= \frac{\pi}{2}\, F\left(-\frac{1}{2},\frac{1}{2},1;k^2\right) = \frac{\pi}{2}\left\{1-\frac{1}{4}\,k^2-\frac{3}{64}\,k^4+\ldots-\left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1}+\ldots\right\}.\end{aligned}$$

Furthermore, for $k^2 \to 1$, they admit the following expansion in the so–called complementary modulus $k' = \sqrt{1-k^2}$:

$$\begin{aligned}\mathbf{K}(k^2) &= \log\frac{4}{k'}+\left(\log\frac{4}{k'}-1\right)\frac{k'^2}{4}+\ldots\,, \\ \mathbf{E}(k^2) &= 1+\left(\log\frac{4}{k'}-\frac{1}{2}\right)\frac{k'^2}{2}+\ldots\,.\end{aligned} \tag{B.3}$$

Note that the complementary elliptic integral of the first kind is defined as

$$\mathbf{K}'(k^2) \;=\; \mathbf{K}(k'^2)\,. \tag{B.4}$$

The function $\mathrm{am}(u,k^2)$, depending on the parameter k, and called Jacobi's elliptic amplitude, is defined through the first order differential equation

$$\left(\frac{d\,\mathrm{am}(u)}{du}\right)^2 \;=\; 1-k^2\sin^2\left[\mathrm{am}(u)\right]\,, \tag{B.5}$$

and it is doubly quasi–periodic in the variable u:

$$\mathrm{am}\left(u+2n\mathbf{K}+2im\mathbf{K}'\right) \;=\; n\pi+\mathrm{am}(u)\,.$$

The Jacobi's elliptic function $\mathrm{sn}(u,k^2)$, defined through the equation

$$\left(\frac{d\,\mathrm{sn}u}{du}\right)^2 \;=\; \left(1-\mathrm{sn}^2u\right)\left(1-k^2\mathrm{sn}^2u\right)\,, \tag{B.6}$$

is related to the amplitude by $\mathrm{sn}\,u = \sin\left(\mathrm{am}\,u\right)$, and it is doubly periodic:

$$\mathrm{sn}\left(u+4n\mathbf{K}+2im\mathbf{K}'\right) \;=\; \mathrm{sn}(u)\,.$$

C Lamé equation

The second order differential equation

$$\left\{\frac{d^2}{du^2}-E-N(N+1)\mathcal{P}(u)\right\}f(u) \;=\; 0\,, \tag{C.1}$$

where E is a real quantity, N is a positive integer and $\mathcal{P}(u)$ denotes the Weierstrass function, is known under the name of N-th Lamé equation. The function $\mathcal{P}(u)$ is a doubly periodic solution of the first order equation (see [30])

$$\left(\frac{d\mathcal{P}}{du}\right)^2 = 4\left(\mathcal{P}-e_1\right)\left(\mathcal{P}-e_2\right)\left(\mathcal{P}-e_3\right) , \tag{C.2}$$

whose characteristic roots e_1, e_2, e_3 uniquely determine the half–periods ω and ω', defined by

$$\mathcal{P}\left(u+2n\omega+2m\omega'\right) = \mathcal{P}(u) .$$

The stability equation (3.7) can be identified with eq. (C.1) for $N=1$, $u=\bar{x}+i\mathbf{K}'$ and $E=\frac{2-k^2}{3}-k^2\bar{\omega}^2$ in virtue of the relation between $\mathcal{P}(u)$ and the Jacobi elliptic function $\mathrm{sn}(u,k)$ (see formulas 8.151 and 8.169 of [30]):

$$k^2\mathrm{sn}^2(\bar{x},k) = \mathcal{P}(\bar{x}+i\mathbf{K}')+\frac{k^2+1}{3} . \tag{C.3}$$

Relation (C.3) holds if the characteristic roots of $\mathcal{P}(u)$ are expressed in terms of k^2 as

$$e_1 = \frac{2-k^2}{3} , \qquad e_2 = \frac{2k^2-1}{3} , \qquad e_3 = -\frac{1+k^2}{3} , \tag{C.4}$$

and, as a consequence, the real and imaginary half periods of $\mathcal{P}(u)$ are given by the elliptic integrals of the first kind

$$\omega = \mathbf{K}(k) , \qquad \omega' = i\mathbf{K}'(k) . \tag{C.5}$$

All the properties of Weierstrass functions that we will use in the following are specified to the case when this identification holds.

In the case $N=1$ the two linearly independent solutions of (C.1) are given by (see [31])

$$f_{\pm a}(u) = \frac{\sigma(u\pm a)}{\sigma(u)}\, e^{\mp u\,\zeta(a)} , \tag{C.6}$$

where a is an auxiliary parameter defined through $\mathcal{P}(a)=E$, and $\sigma(u)$ and $\zeta(u)$ are other kinds of Weierstrass functions:

$$\frac{d\,\zeta(u)}{du} = -\mathcal{P}(u) , \qquad \frac{d\,\log\sigma(u)}{du} = \zeta(u) , \tag{C.7}$$

with the properties

$$\begin{aligned}\zeta(u+2\mathbf{K}) &= \zeta(u)+2\zeta(\mathbf{K}) ,\\ \sigma(u+2\mathbf{K}) &= -\,e^{2(u+\mathbf{K})\zeta(\mathbf{K})}\sigma(u) .\end{aligned} \tag{C.8}$$

As a consequence of eq. (C.8) one obtains the Floquet exponent of $f_{\pm a}(u)$, defined as

$$f(u+2\mathbf{K}) \;=\; f(u)e^{iF(a)}\;, \tag{C.9}$$

in the form

$$F(\pm a) \;=\; \pm 2i\left[\mathbf{K}\,\zeta(a) - a\,\zeta(\mathbf{K})\right]\;. \tag{C.10}$$

The spectrum in the variable E of eq. (C.1) with $N = 1$ is divided in allowed/forbidden bands depending on whether $F(a)$ is real or complex for the corresponding values of a. We have that $E < e_3$ and $e_2 < E < e_1$ correspond to allowed bands, while $e_3 < E < e_2$ and $E > e_1$ are forbidden bands. Note that if we exploit the periodicity of $\mathcal{P}(a)$ and redefine $a \to a' = a + 2n\omega + 2m\omega'$, this only shifts F to $F' = F + 2m\pi$.

The function $\zeta(u)$ admits a series representation [32] that will be very useful for our purposes in Sect. 3.2:

$$\zeta(u) \;=\; \frac{\pi}{2\mathbf{K}}\cot\left(\frac{\pi u}{2\mathbf{K}}\right) + \left(\frac{\mathbf{E}}{\mathbf{K}} + \frac{k^2-2}{3}\right)u + \frac{2\pi}{\mathbf{K}}\sum_{n=1}^{\infty}\frac{h^{2n}}{1-h^{2n}}\sin\left(\frac{n\pi u}{\mathbf{K}}\right)\;, \tag{C.11}$$

where $h = e^{-\pi\mathbf{K}'/\mathbf{K}}$. The small-$k$ expansion of this expression gives

$$\zeta(u) \;=\; \left(\cot u + \frac{u}{3}\right) \;+\; \frac{k^2}{12}\left(u - 3\cot u + 3u\cot^2 u\right) \;+ \tag{C.12}$$

$$+\,\frac{k^4}{64}\left(-3u + (4u^2-5)\cot u + u\cot^2 u + 4u^2\cot^3 u + \sin 2u\right) + \ldots$$

(note that $h \approx \left(\frac{k}{4}\right)^2 + O\left(k^4\right)$). A similar expression takes place for $\mathcal{P}(u)$, by noting that $\mathcal{P}(u) = -\frac{d\,\zeta(u)}{du}$.

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